

CENTRAL AND L^p -CONCENTRATION OF 1-LIPSCHITZ MAPS INTO \mathbb{R} -TREES

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ABSTRACT. In this paper, we study the Lévy-Milman concentration phenomenon of 1-Lipschitz maps from mm-spaces to \mathbb{R} -trees. Our main theorems assert that the concentration to \mathbb{R} -trees is equivalent to the concentration to the real line.

1. INTRODUCTION

This paper is devoted to investigating the Lévy-Milman concentration phenomenon of 1-Lipschitz maps from mm-spaces (metric measure spaces) to \mathbb{R} -trees. Here, an mm-space is a triple (X, d_X, μ_X) of a set X , a complete separable distance function d_X on X , and a finite Borel measure μ_X on (X, d_X) . Let $\{(X_n, d_{X_n}, \mu_{X_n})\}_{n=1}^\infty$ be a sequence of mm-spaces and $\{(Y_n, d_{Y_n})\}_{n=1}^\infty$ a sequence of metric spaces. Given a sequence $\{f_n : X_n \rightarrow Y_n\}_{n=1}^\infty$ of 1-Lipschitz maps, we consider the following three properties:

(i) (Concentration property) There exist points $m_{f_n} \in Y_n$, $n \in \mathbb{N}$, such that

$$\mu_{X_n}(\{x_n \in X_n \mid d_{Y_n}(f_n(x_n), m_{f_n}) \geq \varepsilon\}) \rightarrow 0 \text{ as } n \rightarrow \infty$$

for any $\varepsilon > 0$.

(ii) (Central concentration property) The maps f_n , $n \in \mathbb{N}$, concentrate to the center of mass of the push-forward measure $(f_n)_*(\mu_{X_n})$. In other words, the concentration property (i) holds in the case where m_{f_n} is the center of mass.

(iii) (L^p -concentration property) For a number $p > 0$, we have

$$\int \int_{X_n \times X_n} d_{Y_n}(f_n(x_n), f_n(y_n))^p d\mu_{X_n}(x_n) d\mu_{X_n}(y_n) \rightarrow 0 \text{ as } n \rightarrow \infty$$

Each target metric space Y_n , $n \in \mathbb{N}$, is called a *screen*. Chebyshev's inequality proves that the L^p -concentration (iii) implies the concentration property (i) for any $p > 0$. If each screen Y_n , $n \in \mathbb{N}$, is an Euclidean space \mathbb{R}^k , then the L^p -concentration (iii) for $p \geq 1$ yields the central concentration property (ii) (see Lemma 2.18). The central concentration (ii) is stronger than the concentration property (i). There is an example of maps f_n , $n \in \mathbb{N}$, with the concentration property (i), but not having the central concentration property

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(ii) (see Remark 2.17). In some special cases, the concentration (i) implies the central and L^p -concentration properties (ii) and (iii) (see [3, Subsection 2.4] and [7, Section 3 $\frac{1}{2}$.31]).

Vitali D. Milman first introduced the concentration and the central concentration properties (i) and (ii) for 1-Lipschitz functions (i.e., $Y_n = \mathbb{R}$, $n \in \mathbb{N}$) and emphasized their importance in his investigation of asymptotic geometric analysis (see [11]). Nowadays those properties are widely studied by many literature and blend with various areas of mathematics (see [7], [9], [12], [13], [14], [16], [17] and references therein for further information). M. Gromov first considered the case of general screens in [5], [6], and [7, Chapter 3 $\frac{1}{2}$]. See [3], [4], and [10] for another works of general screens. In [7], Gromov settled the concentration and central concentration properties (i) and (ii) for 1-Lipschitz maps by introducing the *observable diameter* $\text{ObsDiam}_Y(X; -\kappa)$ and the *observable central radius* $\text{ObsCRad}_Y(X; -\kappa)$ for an mm-space X , a metric space Y , and $\kappa > 0$ (see Section 2 for the precise definitions). The L^2 -concentration property (iii) was first appeared in Gromov's paper [5]. Motivated by [5], the author introduced in [3] the *observable L^p -variation* $\text{Obs}L^p\text{-Var}_Y(X)$ to study the property (iii) (see Section 2 for the definition). Note that given a sequence $\{X_n\}_{n=1}^\infty$ of mm-spaces and $\{Y_n\}_{n=1}^\infty$ of metric spaces, $\text{ObsDiam}_{Y_n}(X_n; -\kappa)$ (resp., $\text{ObsCRad}_{Y_n}(X_n; -\kappa)$, $\text{Obs}L^p\text{-Var}_{Y_n}(X_n)$) converges to zero as $n \rightarrow \infty$ for any $\kappa > 0$ if and only if any sequence $\{f_n : X_n \rightarrow Y_n\}_{n=1}^\infty$ of 1-Lipschitz maps (resp., central, L^p -)concentrates.

In this paper, we treat the case of \mathbb{R} -tree screens.

Theorem 1.1. *Let $\{X_n\}_{n=1}^\infty$ be a sequence of mm-spaces. Then, the following (1.1) and (1.2) are equivalent to each other.*

$$(1.1) \quad \text{ObsDiam}_{\mathbb{R}}(X_n; -\kappa) \rightarrow 0 \text{ as } n \rightarrow \infty \text{ for any } \kappa > 0.$$

$$(1.2) \quad \sup\{\text{ObsDiam}_T(X_n; -\kappa) \mid T \text{ is an } \mathbb{R}\text{-tree}\} \rightarrow 0 \text{ as } n \rightarrow \infty \text{ for any } \kappa > 0.$$

Theorem 1.1 is a complete solution to Gromov's exercise in [7, Section 3 $\frac{1}{2}$.32]. In [3, Section 5], the author proved it only for simplicial tree screens. The implication (1.2) \Rightarrow (1.1) is obvious. For the proof of the converse, we define the notion of a *median* for a finite Borel measure on an \mathbb{R} -tree in Section 3 and proves that any 1-Lipschitz maps f_n from X_n into \mathbb{R} -trees concentrate to medians for the push-forward measure $(f_n)_*(\mu_{X_n})$.

To study the central and L^p -concentration for (ii) and (iii) into \mathbb{R} -trees, we estimate the distance between the center of mass and a median of a finite Borel measure on an \mathbb{R} -tree from the above in Section 5. For this estimate, we partially extend K-T. Sturm's characterization of the center of mass on a simplicial tree to the case of an \mathbb{R} -tree (see Proposition 2.12 and Section 4). From the estimate, we bound $\text{ObsCRad}_T(X; -\kappa)$ (resp., $\text{Obs}L^p\text{-Var}_T(X)$) from the above in terms of $\text{ObsCRad}_{\mathbb{R}}(X; -\kappa)$ (resp., $\text{Obs}L^p\text{-Var}_{\mathbb{R}}(X)$) (see Propositions 5.5 and 5.7). As a result, we obtain

Theorem 1.2. *Let $\{X_n\}_{n=1}^\infty$ be a sequence of mm-spaces. Then, the following (1.3) and (1.4) are equivalent to each other.*

$$(1.3) \quad \text{ObsCRad}_{\mathbb{R}}(X_n; -\kappa) \rightarrow 0 \text{ as } n \rightarrow \infty \text{ for any } \kappa > 0.$$

$$(1.4) \quad \sup\{\text{ObsCRad}_T(X_n; -\kappa) \mid T \text{ is an } \mathbb{R}\text{-tree}\} \rightarrow 0 \text{ as } n \rightarrow \infty \text{ for any } \kappa > 0.$$

Theorem 1.3. *Let $\{X_n\}_{n=1}^\infty$ be a sequence of mm-spaces and $p \geq 1$. Then, the following (1.5) and (1.6) are equivalent to each other.*

$$(1.5) \quad \text{Obs}L^p\text{-Var}_{\mathbb{R}}(X_n) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

$$(1.6) \quad \sup\{\text{Obs}L^p\text{-Var}_T(X_n) \mid T \text{ is an } \mathbb{R}\text{-tree}\} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

The condition (1.3) is stronger than (1.1) (see Lemma 2.16 and Remark 2.17), and (1.5) implies (1.3) (see Lemma 2.18). It seems that the conditions (1.3) and (1.5) are not equivalent, but we have no counterexample.

In our previous work, the author investigated the above properties (i), (ii), and (iii) for 1-Lipschitz maps into Hadamard manifolds (see [3, Theorems 1.3, 1.4, and Lemma 4.4]). The L^2 -concentration property (iii) in that case is also studied by Gromov (see [5, Section 13]). Our theorems are thought as of 1-dimensional analogue to these works.

2. PRELIMINARIES

2.1. Basics of the concentration and the L^p -concentration.

2.1.1. *Observable diameter and separation distance.* Let Y be a metric space and ν a Borel measure on Y such that $m := \nu(Y) < +\infty$. We define for any $\kappa > 0$

$$\text{diam}(\nu, m - \kappa) := \inf\{\text{diam } Y_0 \mid Y_0 \subseteq Y \text{ is a Borel subset such that } \nu(Y_0) \geq m - \kappa\}$$

and call it the *partial diameter* of ν .

Definition 2.1 (Observable diameter). Let (X, d_X, μ_X) be an mm-space with $m := \mu_X(X)$ and Y a metric space. For any $\kappa > 0$ we define the *observable diameter* of X by

$$\text{ObsDiam}_Y(X; -\kappa) := \sup\{\text{diam}(f_*(\mu_X), m - \kappa) \mid f : X \rightarrow Y \text{ is a 1-Lipschitz map}\}.$$

The target metric space Y is called the *screen*.

The idea of the observable diameter comes from the quantum and statistical mechanics, that is, we think of μ_X as a state on a configuration space X and f is interpreted as an observable.

Let (X, d_X, μ_X) be an mm-space. For any $\kappa_1, \kappa_2 \geq 0$, we define the *separation distance* $\text{Sep}(X; \kappa_1, \kappa_2) = \text{Sep}(\mu_X; \kappa_1, \kappa_2)$ of X as the supremum of the distance $d_X(A, B)$, where A and B are Borel subsets of X satisfying that $\mu_X(A) \geq \kappa_1$ and $\mu_X(B) \geq \kappa_2$.

The proof of the following lemmas are easy and we omit the proof.

Lemma 2.2 (cf. [7, Section 3 $\frac{1}{2}$.33]). *Let (X, d_X, μ_X) and (Y, d_Y, μ_Y) be two mm-spaces. Assume that a 1-Lipschitz map $f : X \rightarrow Y$ satisfies $f_*(\mu_X) = \mu_Y$. Then we have*

$$\text{Sep}(Y; \kappa_1, \kappa_2) \leq \text{Sep}(X; \kappa_1, \kappa_2)$$

Lemma 2.3. *For any $\kappa > m/2$, we have $\text{Sep}(X; \kappa, \kappa) = 0$.*

The relationships between the observable diameter and the separation distance are the following:

Proposition 2.4 (cf. [7, Section 3 $\frac{1}{2}$.33]). *Let (X, d, μ) be an mm-space and $0 < \kappa' < \kappa$. Then we have*

$$\text{Sep}(X; \kappa, \kappa) \leq \text{ObsDiam}_{\mathbb{R}}(X; -\kappa').$$

Proposition 2.5 (cf. [7, Section 3 $\frac{1}{2}$.33]). *For any $\kappa > 0$, we have*

$$\text{ObsDiam}_{\mathbb{R}}(X; -2\kappa) \leq \text{Sep}(X; \kappa, \kappa).$$

See [4, Subsection 2.2] for details of the proofs of the above propositions.

Corollary 2.6 (cf. [7, Section 3 $\frac{1}{2}$.33]). *A sequence $\{X_n\}_{n=1}^{\infty}$ of mm-spaces satisfies that*

$$\text{ObsDiam}_{\mathbb{R}}(X_n; -\kappa) \rightarrow 0 \text{ as } n \rightarrow \infty$$

for any $\kappa > 0$ if and only if $\text{Sep}(X_n; \kappa, \kappa) \rightarrow 0$ as $n \rightarrow \infty$ for any $\kappa > 0$.

2.1.2. Observable L^p -variation. Let (X, d_X, μ_X) be an mm-space and (Y, d_Y) a metric space. Given a Borel measure ν on Y and $p \in (0, +\infty)$, we put

$$V_p(\nu) := \left(\int \int_{Y \times Y} d_Y(x, y)^p d\nu(x) d\nu(y) \right)^{1/p}.$$

For a Borel measurable map $f : X \rightarrow Y$, we also put $V_p(f) := V_p(f_*(\mu_X))$

Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of mm-spaces and $\{Y_n\}_{n=1}^{\infty}$ a sequence of metric spaces. For any $p \in (0, +\infty]$, we say that a sequence $\{f_n : X_n \rightarrow Y_n\}_{n=1}^{\infty}$ of Borel measurable maps L^p -concentrates if $V_p(f_n) \rightarrow 0$ as $n \rightarrow \infty$.

Given an mm-space X and a metric space Y we define

$$\text{Obs}L^p\text{-Var}_Y(X) := \sup\{V_p(f) \mid f : X \rightarrow Y \text{ is a 1-Lipschitz map}\},$$

and call it the *observable L^p -variation* of X .

Lemma 2.7. *For any closed subset $A \subset X$, we have*

$$\text{Obs}L^p\text{-Var}_{\mathbb{R}}(A) \leq \text{Obs}L^p\text{-Var}_{\mathbb{R}}(X).$$

Proof. Let $f : A \rightarrow \mathbb{R}$ be an arbitrary 1-Lipschitz function. From [1, Theorem 3.1.2], there exists a 1-Lipschitz extension of f , say $\tilde{f} : X \rightarrow \mathbb{R}$. Hence, we get

$$V_p(f) \leq V_p(\tilde{f}) \leq \text{Obs}L^p\text{-Var}_{\mathbb{R}}(X).$$

This completes the proof. \square

See [3, Subsection 2.4] for the relationships between the observable diameter and the observable L^p -variation.

2.2. Basics of \mathbb{R} -trees. Before reviewing the definition of \mathbb{R} -trees, we recall some standard terminologies in metric geometry. Let (X, d_X) be a metric space. A rectifiable curve $\eta : [0, 1] \rightarrow X$ is called a *geodesic* if its arclength coincides with the distance $d_X(\eta(0), \eta(1))$ and it has a constant speed, i.e., parameterized proportionally to the arc length. We say that (X, d_X) is a *geodesic space* if any two points in X are joined by a geodesic between them. Let X be a geodesic space. A *geodesic triangle* in X is the union of the image of three geodesics joining a triple of points in X pairwise. A subset $A \subseteq X$ is called *convex* if every geodesic joining two points in A is contained in A .

A complete metric space (T, d_T) is called an \mathbb{R} -tree if it has the following properties:

- (1) For all $z, w \in T$ there exists a unique unit speed geodesic $\phi_{z,w}$ from z to w .
- (2) The image of every simple path in T is the image of a geodesic.

Denote by $[z, w]_T$ the image of the geodesic $\phi_{z,w}$. We also put $(z, w)_T := [z, w]_T \setminus \{z\}$ and $(z, w)_T := [z, w]_T \setminus \{z, w\}$. A complete geodesic space T is an \mathbb{R} -tree if and only if it is 0-hyperbolic, that is to say, every edge in any geodesic triangle in T is included in the union of the other two edges. See [2] for another characterizations of \mathbb{R} -trees. Given $z \in T$, we indicate by $\mathcal{C}_T(z)$ the set of all connected components of $T \setminus \{z\}$. We also denote by $\mathcal{C}'_T(z)$ the set of all $\{z\} \cup T'$ for $T' \in \mathcal{C}_T(z)$. Although the following lemma is somewhat standard, we prove it for the completeness.

Lemma 2.8. *Each $T' \in \mathcal{C}_T(z)$ is convex.*

Proof. From the property (2) of \mathbb{R} -trees, it is sufficient to prove that T' is arcwise connected. Taking a point $z \in T'$, we put

$$A := \{w \in T' \mid z \text{ and } w \text{ are connected by a path in } T'\}.$$

It is easy to see that the set A is closed in T' . Since every metric ball in T is arcwise connected, the set A is also open. Since T' is connected, we get $T' = A$. This completes the proof. \square

A subset in an \mathbb{R} -tree is called a *subtree* if it is a closed convex subset. Note that a subtree is itself an \mathbb{R} -tree.

Proposition 2.9. *Every connected subset in an \mathbb{R} -tree is convex.*

Proof. Let T be an \mathbb{R} -tree. Suppose that there exists a connected subset $T' \subseteq T$ which is not convex. Then, there are points $z, w \in T'$ and $\tilde{z} \in (z, w)_T$ such that $\tilde{z} \notin T'$. Since $T' = \bigcup \{T' \cap C \mid C \in \mathcal{C}_T(\tilde{z})\}$ and each $C \in \mathcal{C}_T(\tilde{z})$ is open, from the connectivity of T' , there is $C_0 \in \mathcal{C}_T(\tilde{z})$ such that $T' \subseteq C_0$. Since C_0 is convex by Lemma 2.8, we get $\tilde{z} \in [z, w]_T \subseteq C_0$. This is a contradiction since $\tilde{z} \notin C_0$. This completes the proof. \square

2.3. Center of mass of a measure on a CAT(0)-space and observable central radius.

2.3.1. *Basics of the center of mass of a measure on CAT(0)-spaces.* In this subsection, we review Sturm's works about measures on a CAT(0)-spaces. Refer [8] and [15] for details. A geodesic metric space X is called a CAT(0)-space if we have

$$d_X(x, \gamma(1/2))^2 \leq \frac{1}{2} d_X(x, y)^2 + \frac{1}{2} d_X(x, z)^2 - \frac{1}{4} d_X(y, z)^2$$

for any $x, y, z \in X$ and any minimizing geodesic $\gamma : [0, 1] \rightarrow X$ from y to z . For example, Hadamard manifolds, Hilbert spaces, and \mathbb{R} -trees are all CAT(0)-spaces.

Let (X, d_X) be a metric space. We denote by $\mathcal{B}(X)$ the set of all finite Borel measures ν on X with the separable support. We indicate by $\mathcal{B}^1(X)$ the set of all Borel measures $\nu \in \mathcal{B}(X)$ such that $\int_X d_X(x, y) d\nu(y) < +\infty$ for some (hence all) $x \in X$. We also indicate by $\mathcal{P}^1(X)$ the set of all probability measures in $\mathcal{B}^1(X)$. For any $\nu \in \mathcal{B}^1(X)$ and $z \in X$, we consider the function $h_{z, \nu} : X \rightarrow \mathbb{R}$ defined by

$$h_{z, \nu}(x) := \int_X \{d_X(x, y)^2 - d_X(z, y)^2\} d\nu(y).$$

Note that

$$\int_X |d_X(x, y)^2 - d_X(z, y)^2| d\nu(y) \leq d_X(x, z) \int_X \{d_X(x, y) + d_X(z, y)\} d\nu(y) < +\infty.$$

A point $z_0 \in X$ is called the *center of mass* of the measure $\nu \in \mathcal{B}^1(X)$ if for any $z \in X$, z_0 is a unique minimizing point of the function $h_{z, \nu}$. We denote the point z_0 by $c(\nu)$. A metric space X is said to be *centric* if every $\nu \in \mathcal{B}^1(X)$ has the center of mass.

Proposition 2.10 (cf. [15, Proposition 4.3]). *A CAT(0)-space is centric.*

A simple variational argument yields the following lemma.

Lemma 2.11 (cf. [15, Proposition 5.4]). *Let H be a Hilbert space. Then for each $\nu \in \mathcal{B}^1(H)$ with $m = \nu(X)$, we have*

$$c(\nu) = \frac{1}{m} \int_H y d\nu(y).$$

Let (T, d_T) be an \mathbb{R} -tree and $\nu \in \mathcal{B}^1(T)$. For $z \in T$ and $T' \in \mathcal{C}'_T(z)$, we put

$$c_{z, T'}(\nu) := \int_{T'} d_T(z, w) d\nu(w) - \int_{T \setminus T'} d_T(z, w) d\nu(w).$$

Let us consider a (possibly infinite) simplicial tree T_s . Here, the length of each edge of T_s is not necessarily equal to 1. We assume that every vertex of T_s is an isolated point in the vertex set of T_s .

Proposition 2.12 (cf. [15, Proposition 5.9]). *Let $\nu \in \mathcal{B}^1(T_s)$ and $z \in T_s$. Then, $z = c(\nu)$ if and only if $c_{z, T'}(\nu) \leq 0$ for any $T' \in \mathcal{C}'_{T_s}(z)$.*

Proposition 2.13 (cf. [15, Proposition 6.1]). *Let N be a CAT(0)-space and $\nu \in \mathcal{B}^1(N)$. Assume that the support of ν is contained in a closed convex subset K of N . Then, we have $c(\nu) \in K$.*

Let X be a metric space. For $\mu, \nu \in \mathcal{P}^1(X)$, we define the L^1 -Wasserstein distance $d_1^W(\mu, \nu)$ between μ and ν as the infimum of $\int_{X \times X} d_X(x, y) d\pi(x, y)$, where $\pi \in \mathcal{P}^1(X \times X)$ runs over all *couplings* of μ and ν , that is, the measures π with the property that $\pi(A \times X) = \mu(A)$ and $\pi(X \times A) = \nu(A)$ for any Borel subset $A \subseteq X$.

Lemma 2.14 (cf. [18, Theorem 7.12]). *A sequence $\{\mu_n\}_{n=1}^\infty \subseteq \mathcal{P}^1(X)$ converges to $\mu \in \mathcal{P}^1(X)$ with respect to the distance function d_1^W if and only if the sequence $\{\mu_n\}_{n=1}^\infty$ converges weakly to the measure μ and*

$$\lim_{n \rightarrow \infty} \int_X d_X(x, y) d\mu_n(y) = \int_X d_X(x, y) d\mu(y)$$

for some (and then any) $x \in X$.

Theorem 2.15 (cf. [15, Theorem 6.3]). *Let N be a $CAT(0)$ -space. Given $\mu, \nu \in \mathcal{P}^1(N)$, we have $d_N(c(\mu), c(\nu)) \leq d_1^W(\mu, \nu)$.*

2.3.2. Observable central radius. Let Y be a metric space and assume that $\nu \in \mathcal{B}^1(Y)$ has the center of mass. We denote by $B_Y(y, r)$ the closed ball in Y centered at $y \in Y$ and with radius $r > 0$. For any $\kappa > 0$, putting $m := \nu(Y)$, we define the *central radius* $\text{CRad}(\nu, m - \kappa)$ of ν as the infimum of $\rho > 0$ such that $\nu(B_Y(c(\nu), \rho)) \geq m - \kappa$.

Let (X, d_X, μ_X) be an mm-space with $\mu_X \in \mathcal{B}^1(X)$ and Y a centric metric space. For any $\kappa > 0$, we define

$$\text{ObsCRad}_Y(X; -\kappa) := \sup\{\text{CRad}(f_*(\mu_X), m - \kappa) \mid f : X \rightarrow Y \text{ is a 1-Lipschitz map}\},$$

and call it the *observable central radius* of X .

Lemma 2.16 (cf. [7, Section 3 $\frac{1}{2}$.31]). *For any $\kappa > 0$, we have*

$$\text{diam}(\nu, m - \kappa) \leq 2 \text{CRad}(\nu, m - \kappa).$$

In particular, we get

$$\text{ObsDiam}_Y(X; -\kappa) \leq 2 \text{ObsCRad}_Y(X; -\kappa).$$

Remark 2.17. From the above lemma, we see that the central concentration implies the concentration. The converse is not true in general. For example, consider the mm-spaces $X_n := \{x_n, y_n\}$ with distance function d_{X_n} given by $d_{X_n}(x_n, y_n) := n$ and with a Borel probability measure μ_{X_n} given by $\mu_{X_n}(\{x_n\}) := 1 - 1/n$ and $\mu_{X_n}(\{y_n\}) := 1/n$. Then, 1-Lipschitz maps $f_n : X_n \rightarrow \mathbb{R}$ defined by $f_n(x) := d_{X_n}(x, x_n)$ satisfy that

$$(f_n)_*(\mu_{X_n})(B_{\mathbb{R}}(c((f_n)_*(\mu_{X_n})), 1/2)) = 0$$

for any $n \in \mathbb{N}$, whereas $\text{ObsDiam}_{\mathbb{R}}(X_n; -\kappa) \rightarrow 0$ as $n \rightarrow \infty$.

Lemma 2.18. *Let $\nu \in \mathcal{B}^1(\mathbb{R}^n)$ with $m := \nu(\mathbb{R}^n)$. Then, for any $p \geq 1$ and $\kappa > 0$, we have*

$$(2.1) \quad \text{CRad}(\nu, m - \kappa) \leq \frac{V_p(\nu)}{(m\kappa)^{1/p}}.$$

In the case of $p = 2$, we also have the better estimate

$$(2.2) \quad \text{CRad}(\nu, m - \kappa) \leq \frac{V_2(\nu)}{\sqrt{2m\kappa}}.$$

Proof. We shall prove that $\nu(\mathbb{R}^n \setminus B_{\mathbb{R}^n}(c(\nu), \rho_0)) \leq \kappa$ for $\rho_0 := V_p(\nu)/(m\kappa)^{1/p}$. Suppose that $\nu(\mathbb{R}^n \setminus B_{\mathbb{R}^n}(c(\nu), \rho_0)) > \kappa$. From Lemma 2.11, we get

$$\int_{\mathbb{R}^n} |c(\nu) - x|^p d\nu(x) \leq \frac{V_p(\nu)^p}{m}.$$

Hence, from Chebyshev's inequality, we see that

$$\frac{V_p(\nu)^p}{m} = \rho_0^p \kappa < \int_{\mathbb{R}^n} |c(\nu) - x|^p d\nu(x) \leq \frac{V_p(\nu)^p}{m},$$

which is a contradiction. Therefore, we obtain $\nu(B_{\mathbb{R}^n}(c(\nu), \rho_0)) \geq m - \kappa$ and so (2.1).

Since

$$\int_{\mathbb{R}^n} |c(\nu) - x|^2 d\nu(x) = \frac{V_2(\nu)^2}{2m},$$

the same argument yields (2.2). This completes the proof. \square

Corollary 2.19. *Let X be an mm-space with $\mu_X \in \mathcal{B}^1(X)$. Then, for any $p \geq 1$, we have*

$$(2.3) \quad \text{ObsCRad}_{\mathbb{R}^n}(X; -\kappa) \leq \frac{1}{(m\kappa)^{1/p}} \text{Obs}L^p\text{-Var}_{\mathbb{R}^n}(X).$$

In the case of $p = 2$, we also have the better estimate

$$(2.4) \quad \text{ObsCRad}_{\mathbb{R}^n}(X; -\kappa) \leq \frac{1}{\sqrt{2m\kappa}} \text{Obs}L^2\text{-Var}_{\mathbb{R}^n}(X)$$

Corollary 2.20. *Let X be an mm-space. Then, for any $p \geq 1$ and $\kappa > 0$, we have*

$$(2.5) \quad \text{Sep}(X; \kappa, \kappa) \leq \frac{2}{(m\kappa)^{1/p}} \text{Obs}L^p\text{-Var}_{\mathbb{R}}(X).$$

In the case of $p = 2$, we also have

$$(2.6) \quad \text{Sep}(X; \kappa, \kappa) \leq \sqrt{\frac{2}{m\kappa}} \text{Obs}L^2\text{-Var}_{\mathbb{R}}(X).$$

Proof. Assume first that there is a 1-Lipschitz function $f : X \rightarrow \mathbb{R}$ such that $f_*(\mu_X) \notin \mathcal{B}^1(\mathbb{R})$. From Hölder's inequality, we have $\int_{\mathbb{R}} |x - y|^p df_*(\mu_X)(y) = +\infty$ for any $x \in X$. This implies $V_p(f) = +\infty$ and so $\text{Obs}L^p\text{-Var}_{\mathbb{R}}(X) = +\infty$.

We consider the other case that $f_*(\mu_X) \in \mathcal{B}^1(\mathbb{R})$ for any 1-Lipschitz function $f : X \rightarrow \mathbb{R}$. Combining Proposition 2.4 with Lemma 2.16 and (2.3), we have

$$\text{Sep}(X; \kappa, \kappa) \leq \frac{2}{(m\kappa')^{1/p}} \text{Obs}L^p\text{-Var}_{\mathbb{R}}(X)$$

for any $\kappa > \kappa' > 0$. Letting $\kappa' \rightarrow \kappa$, we have (2.5). Replacing (2.3) with (2.4) in the above argument, we also obtain (2.6). \square

3. EXISTENCE OF A MEDIAN ON AN \mathbb{R} -TREE

Let T be an \mathbb{R} -tree and ν a finite Borel measure on T with $m := \nu(T) < +\infty$. A median of ν is a point $z \in T$ such that there exist two subtrees $T', T'' \subseteq T$ such that $T = T' \cup T''$, $T' \cap T'' = \{z\}$, $\nu(T') \geq m/3$, and $\nu(T'') \geq m/3$. The existence of a median of a finite Borel measure on a simplicial tree is proved in [3, Proposition 5.2]. The purpose of this section is to prove the existence of a median of a finite Borel measure on an \mathbb{R} -tree, which is needed for the proofs of our main theorems. Although the proof of the existence is similar to the proof for the case of a simplicial tree, we prove it for the completeness:

Proposition 3.1. *Every finite Borel measure on an \mathbb{R} -tree has a median.*

Proof. Let ν be a finite Borel measure on an \mathbb{R} -tree with $m := \nu(T)$. Assume that a point $z \in T$ satisfies that $\nu(T') < m/3$ for any $T' \in \mathcal{C}'_T(z)$, then it is easy to check that z is a median of ν . So, we assume that for any $z \in T$ there exists $T(z) \in \mathcal{C}'_T(z)$ such that $\nu(T(z)) \geq m/3$. If for some $z \in T$, there exists $T' \in \mathcal{C}'_T(z) \setminus \{T(z)\}$ such that $\nu(T') \geq m/3$, then this z is a median of ν . Thereby, we also assume that $\nu(T') < m/3$ for any $z \in T$ and $T' \in \mathcal{C}'_T(z) \setminus \{T(z)\}$.

Fixing a point $z_0 \in T$, we assume that there exists $z \in T(z_0) \setminus \{z_0\}$ such that $z_0 \in T(z)$. Put

$$t_0 := \inf\{t \in (0, d_T(z_0, z)) \mid z_0 \in T(\phi_{z_0, z}(t))\}.$$

Claim 3.2. $\phi_{z_0, z}(t_0)$ is a median of ν .

Proof. Assume first that $t_0 = 0$. Then, taking a monotone decreasing sequence $\{t_n\}_{n=1}^\infty \subseteq (0, d_T(z_0, z)]$ such that $t_n \rightarrow 0$ as $n \rightarrow \infty$ and $z_0 \in T(\phi_{z_0, z}(t_n))$ for any $n \in \mathbb{N}$, we shall show that $\bigcap_{n=1}^\infty T(\phi_{z_0, z}(t_n)) \subseteq (T \setminus T(z_0)) \cup \{z_0\}$. If it is, we conclude that the point $z_0 = \phi_{z_0, z}(0)$ is a median of ν as follows: From the uniqueness of $T(\phi_{z_0, z}(t_n))$, we have $T(\phi_{z_0, z}(t_{n+1})) \subseteq T(\phi_{z_0, z}(t_n))$ for each $n \in \mathbb{N}$. Thus, we get $\nu(\bigcap_{n=1}^\infty T(\phi_{z_0, z}(t_n))) = \lim_{n \rightarrow \infty} \nu(T(\phi_{z_0, z}(t_n))) \geq m/3$.

Suppose that there exists $w \in T(z_0) \setminus \{z_0\} \cap \bigcap_{n=1}^\infty T(\phi_{z_0, z}(t_n))$. Note that $(z_0, z]_T \cap (z_0, w]_T \neq \emptyset$. Actually, suppose that $(z_0, z]_T \cap (z_0, w]_T = \emptyset$. Then, it follows from the property (2) of \mathbb{R} -trees that $[z, w]_T = [z_0, z]_T \cup [z_0, w]_T$. Especially, we have $z_0 \in [z, w]_T$. Since $T(z_0) \setminus \{z_0\}$ is convex by virtue of Lemma 2.8, $[z, w]_T$ does not contain the point z_0 . This is a contradiction. Thus, there exists $t \in (0, d_T(z_0, z)]$ such that $\phi_{z_0, z}(t) \in (z_0, z]_T \cap (z_0, w]_T$. We pick $n_0 \in \mathbb{N}$ with $t_{n_0} < t$. Since $w \in T(z_0) \setminus \{z_0\} \cap \bigcap_{n=1}^\infty T(\phi_{z_0, z}(t_n)) \subseteq T(\phi_{z_0, z}(t_{n_0})) \setminus \{z_0\}$, we get $\phi_{z_0, z}(t) \in (z_0, w]_T \subseteq T(\phi_{z_0, z}(t_{n_0})) \setminus \{z_0\}$. Thereby, we get $\phi_{z_0, z}(t) \in T(\phi_{z_0, z}(t_{n_0})) \setminus \{\phi_{z_0, z}(t_{n_0})\}$. Therefore, since $z_0 \in T(\phi_{z_0, z}(t_{n_0})) \setminus \{\phi_{z_0, z}(t_{n_0})\}$ and $T(\phi_{z_0, z}(t_{n_0})) \setminus \{\phi_{z_0, z}(t_{n_0})\}$ is convex, we obtain

$$\phi_{z_0, z}(t_n) \in [z_0, \phi_{z_0, z}(t)]_T \subseteq T(\phi_{z_0, z}(t_n)) \setminus \{\phi_{z_0, z}(t_n)\}.$$

This is a contradiction. Therefore, we have $\bigcap_{n=1}^\infty T(\phi_{z_0, z}(t_n)) \subseteq (T \setminus T(z_0)) \cup \{z_0\}$.

We consider the other case that $t_0 > 0$. Take a monotone increasing sequence $\{t_n\}_{n=1}^\infty \subseteq (0, +\infty)$ such that $t_n \rightarrow t_0$ as $n \rightarrow \infty$ and $z_0 \notin T(\phi_{z_0, z}(t_n))$ for each $n \in \mathbb{N}$. Then,

the same proof in the case of $t_0 = 0$ implies that $\nu(\bigcap_{n=1}^{\infty} T(\phi_{z_0,z}(t_n))) \geq m/3$ and $\bigcap_{n=1}^{\infty} T(\phi_{z_0,z}(t_n)) \subseteq (T \setminus T(\phi_{z_0,z}(t_0))) \cup \{\phi_{z_0,z}(t_0)\}$. Therefore, $\phi_{z_0,z}(t_0)$ is a median of ν . This completes the proof of the claim. \square

We next assume that $z_0 \notin T(z)$ for any $z \in T(z_0)$. We denote by Γ the set of all unit speed geodesics $\gamma : [0, L(\gamma)] \rightarrow T(z_0)$ such that $\gamma(0) = z_0$ and $\gamma([t, L(\gamma)]) \subseteq T(\gamma(t))$ for any $t \in [0, L(\gamma)]$. Because of the assumption, we easily see

Claim 3.3. *For any $z \in T(z_0)$, we have $\phi_{z_0,z} \in \Gamma$.*

Claim 3.4. *For any $\gamma, \gamma' \in \Gamma$ with $L(\gamma) \leq L(\gamma')$, we have*

$$[\gamma(0), \gamma(L(\gamma))]_T \subseteq [\gamma'(0), \gamma'(L(\gamma'))]_T.$$

Proof. Suppose that

$$t_0 := \sup\{t \in [0, L(\gamma)] \mid [\gamma(0), \gamma(t)]_T \subseteq [\gamma'(0), \gamma'(L(\gamma'))]_T\} < L(\gamma).$$

Then, we have $\gamma(t) \notin [\gamma'(0), \gamma'(L(\gamma'))]_T$ for any $t > t_0$. Actually, if $\gamma(t) \in [\gamma'(0), \gamma'(L(\gamma'))]_T$, then we have $\gamma(t) = \gamma'(t)$. Thus, $[\gamma(t_0), \gamma(t)]_T = [\gamma'(t_0), \gamma'(t)]_T$ by the property (2) of the \mathbb{R} -trees. Thereby, we get $[\gamma(0), \gamma(t)]_T \subseteq [\gamma'(0), \gamma'(L(\gamma'))]_T$. Since $t > t_0$, this contradicts the definition of t_0 . Therefore, from the property (2) of \mathbb{R} -trees, we have

$$(3.1) \quad [\gamma(L(\gamma)), \gamma'(L(\gamma'))]_T = [\gamma(t_0), \gamma(L(\gamma))]_T \cup [\gamma'(t_0), \gamma'(L(\gamma'))]_T.$$

Since $\gamma, \gamma' \in \Gamma$, we have $\gamma(L(\gamma)), \gamma'(L(\gamma')) \in T(\gamma(t_0)) \setminus \{\gamma(t_0)\}$. So, from the convexity of $T(\gamma(t_0)) \setminus \{\gamma(t_0)\}$, we get $[\gamma(L(\gamma)), \gamma'(L(\gamma'))]_T \subseteq T(\gamma(t_0)) \setminus \{\gamma(t_0)\}$. This is a contradiction, because $\gamma(t_0) \in [\gamma(L(\gamma)), \gamma'(L(\gamma'))]_T$ from (3.1). This completes the proof of the claim. \square

Putting $\alpha := \sup\{L(\gamma) \mid \gamma \in \Gamma\}$, we shall show that $\alpha < +\infty$. If $\alpha < +\infty$, we finish the proof of the proposition as follows: From the completeness of \mathbb{R} -trees and Claim 3.4, there exists a unique $\gamma \in \Gamma$ with $L(\gamma) = \alpha$. We also note that $\alpha > 0$ by Claim 3.3. Thus, there exists a monotone increasing sequence $\{t_n\}_{n=1}^{\infty}$ of positive numbers such that $t_n \rightarrow \alpha$ as $n \rightarrow \infty$. We easily see that $T(\gamma(t_{n+1})) \subseteq T(\gamma(t_n))$ for any $n \in \mathbb{N}$ and $\bigcap_{n=1}^{\infty} T(\gamma(t_n)) = \{\gamma(L(\gamma))\}$. Since $\nu(T(\gamma(t_n))) \geq m/3$, the point $\gamma(L(\gamma))$ is a median of ν .

Suppose that $\alpha = +\infty$. Then, taking a sequence $\{\gamma_n\}_{n=1}^{\infty} \subseteq \Gamma$ such that $L(\gamma_n) < L(\gamma_{n+1})$ for any $n \in \mathbb{N}$ and $L(\gamma_n) \rightarrow +\infty$ as $n \rightarrow \infty$, we obtain $\bigcap_{n=1}^{\infty} T(\gamma_n(L(\gamma_n))) = \emptyset$. Since $T(\gamma_n(L(\gamma_n))) \subseteq T(\gamma_{n+1}(L(\gamma_{n+1})))$ for any $n \in \mathbb{N}$, we have

$$0 = \nu\left(\bigcap_{n=1}^{\infty} T(\gamma_n(L(\gamma_n)))\right) = \lim_{n \rightarrow \infty} \nu(T(\gamma_n(L(\gamma_n)))) \geq \frac{m}{3},$$

which is a contradiction. This completes the proof of the proposition. \square

4. THE NECESSITY OF PROPOSITION 2.12 FOR \mathbb{R} -TREES

In order to prove the main theorems, we extend the necessity of Proposition 2.12 for \mathbb{R} -trees:

Proposition 4.1. *Let T be an \mathbb{R} -tree and $\nu \in \mathcal{B}^1(T)$. Then, we have $c_{c(\nu), T'}(\nu) \leq 0$ for any $T' \in \mathcal{C}'_T(c(\nu))$.*

Proof. For simplicities, we assume that $\nu(T) = 1$. We shall approximate the measure ν by a measure whose support lies on a simplicial tree in T . Given $n \in \mathbb{N}$, there exists a compact subset $K_n \subseteq T$ such that $\nu(T \setminus K_n) < 1/n$ and $\int_{T \setminus K_n} d_T(c(\nu), w) d\nu(w) < 1/n$. Take a $(1/n)$ -net $\{z_i^n\}_{i=1}^{l_n}$ of K_n with mutually different elements such that $d_T(c(\nu), z_1^n) < 1/n$. We then take a sequence $\{A_i^n\}_{i=1}^{l_n}$ of mutually disjoint Borel subset of K_n such that $z_i^n \in A_i^n$, $\text{diam } A_i^n \leq 1/n$, and $K_n = \bigcup_{i=1}^{l_n} A_i^n$. Define the Borel probability measure ν_n on $\{z_i^n\}_{i=1}^{l_n}$ by $\nu(\{z_1^n\}) := \nu(A_1^n) + \nu(T \setminus K_n)$ and $\nu(\{z_i^n\}) := \nu(A_i^n)$ for $i \geq 2$.

Claim 4.2. $d_1^W(\nu_n, \nu) \rightarrow 0$ as $n \rightarrow \infty$.

Proof. We shall show that

$$(4.1) \quad \lim_{n \rightarrow \infty} \int_T d_T(c(\nu), w) d\nu_n(w) = \int_T d_T(c(\nu), w) d\nu(w).$$

Since

$$\int_T d_T(c(\nu), w) d\nu_n(w) = \sum_{i=1}^{l_n} d_T(c(\nu), z_i^n) \nu(A_i^n) + d_T(c(\nu), z_1^n) \nu(T \setminus K_n),$$

we have

$$(4.2) \quad \left| \int_T d_T(c(\nu), w) d\nu_n(w) - \sum_{i=1}^{l_n} d_T(c(\nu), z_i^n) \nu(A_i^n) \right| < \frac{1}{n}.$$

From $\text{diam } A_i^n < 1/n$, we get

$$(4.3) \quad \left| \sum_{i=1}^{l_n} d_T(c(\nu), z_i^n) \nu(A_i^n) - \int_{K_n} d_T(c(\nu), w) d\nu(w) \right| \\ = \left| \sum_{i=1}^{l_n} \int_{A_i^n} \{d_T(c(\nu), w) - d_T(c(\nu), z_i^n)\} d\nu(w) \right| \leq \sum_{i=1}^{l_n} \int_{A_i^n} d_T(w, z_i^n) d\nu(w) < \frac{1}{n}.$$

Hence, combining (4.2) with (4.3) and

$$\left| \int_{K_n} d_T(c(\nu), w) d\nu(w) - \int_T d_T(c(\nu), w) d\nu(w) \right| \leq \int_{T \setminus K_n} d_T(c(\nu), w) d\nu(w) < \frac{1}{n},$$

we obtain (4.1). The same way of the above proof shows that the sequence $\{\nu_n\}_{n=1}^\infty$ converges weakly to the measure ν . Therefore, by using Lemma 2.14, this completes the proof of the claim. \square

Applying Claim 4.2 to Theorem 2.15, we get $c(\nu_n) \rightarrow c(\nu)$ as $n \rightarrow \infty$. Since the convex hull in T of the set $\{z_i^n\}_{i=1}^{l_n}$ is a simplicial tree with finite vertex set and $c(\nu_n)$ is contained in the convex hull by Proposition 2.13, it follows from Proposition 2.12 that $c_{\tilde{T}, c(\nu_n)}(\nu_n) \leq 0$ for any $\tilde{T} \in \mathcal{C}'_T(c(\nu_n))$. Let $T' \in \mathcal{C}'_T(c(\nu))$.

Assume first that $c(\nu_n) \in T \setminus T'$ for infinitely many $n \in \mathbb{N}$. Then, taking $T_n \in \mathcal{C}'_T(c(\nu_n))$ with $T' \subseteq T_n$, we have

$$c_{T', c(\nu)}(\nu_n) \leq c_{T_n, c(\nu_n)}(\nu_n) + d_T(c(\nu_n), c(\nu)) \leq d_T(c(\nu_n), c(\nu)).$$

Therefore, we obtain $c_{T', c(\nu)}(\nu) = \lim_{n \rightarrow \infty} c_{T', c(\nu)}(\nu_n) \leq 0$.

We consider the other case that $c(\nu_n) \in T'$ for any $n \in \mathbb{N}$. Let $z_n \in [c(\nu), c(\nu_1)]_T$ be the unique point such that

$$d_T(z_n, c(\nu_n)) = \inf\{d_T(z, c(\nu_n)) \mid z \in [c(\nu), c(\nu_1)]_T\}.$$

By taking a subsequence, we may assume that $d_T(c(\nu), z_{n+1}) \leq d_T(c(\nu), z_n)$ for any $n \in \mathbb{N}$. For each $n \geq 2$, we take $T_n \in \mathcal{C}'_T(z_n)$ and $\tilde{T}_n \in \mathcal{C}'_T(c(\nu_n))$ such that $c(\nu_1) \in T_n$ and $c(\nu_1) \in \tilde{T}_n$. Observe that $T_n \subseteq T_{n+1}$. Since $T_n \subseteq \tilde{T}_n$, we have

$$(4.4) \quad c_{T_n, z_n}(\nu_n) \leq c_{\tilde{T}_n, c(\nu_n)}(\nu_n) + d_T(c(\nu_n), z_n) \leq d_T(c(\nu_n), z_n).$$

We also easily see

Claim 4.3. $T' \setminus \{c(\nu)\} = \bigcup_{n=2}^{\infty} T_n$.

The same proof of Claim 4.2 implies that

$$\sup \left\{ \left| \int_A d_T(z_n, w) d\nu_n(w) - \int_A d_T(z_n, w) d\nu(w) \right| \mid A \subseteq T \text{ is a Borel subset} \right\} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Combining this with (4.4) and Claim 4.3, we obtain

$$c_{T', c(\nu)}(\nu) = \lim_{n \rightarrow \infty} c_{T_n, z_n}(\nu) = \lim_{n \rightarrow \infty} c_{T_n, z_n}(\nu_n) \leq \lim_{n \rightarrow \infty} d_T(c(\nu_n), z_n) = 0.$$

This completes the proof of the proposition. \square

The author does not know whether the converse of Proposition 4.1 holds or not.

5. PROOF OF THE MAIN THEOREMS

Combining Proposition 3.1 with the same proof of [3, Lemma 5.3] implies the following proposition:

Proposition 5.1. *Let T be an \mathbb{R} -tree and ν a finite Borel measure. Then, for any $\kappa > 0$, we have*

$$(5.1) \quad \nu \left(B_T \left(m_\nu, \text{Sep} \left(\nu; \frac{m}{3}, \frac{\kappa}{2} \right) \right) \right) \geq m - \kappa,$$

where m_ν is a median of the measure ν . In particular, letting X be an mm-space, we have

$$(5.2) \quad \text{ObsDiam}_T(X; -\kappa) \leq 2 \text{Sep} \left(X; \frac{m}{3}, \frac{\kappa}{2} \right).$$

Proposition 5.1 together with Corollary 2.6 yields Theorem 1.1. The following way to prove Theorem 1.1 is much easier and more straightforward than the above way, that is, to prove the existence of a median of a measure on \mathbb{R} -trees.

Proof of Theorem 1.1. Our goal is to prove the following inequality:

$$(5.3) \quad \text{ObsDiam}_T(X; -\kappa) \leq 2 \text{Sep} \left(X; \frac{\kappa}{3}, \frac{\kappa}{3} \right) + 4 \text{ObsDiam}_{\mathbb{R}}(X; -\kappa)$$

for any $\kappa > 0$. Let $f : X \rightarrow T$ be an arbitrary 1-Lipschitz map. Fixing a point $z_0 \in T$, we shall consider the function $g : T \rightarrow \mathbb{R}$ defined by $g(z) := d_T(z, z_0)$. Since $g \circ f : X \rightarrow \mathbb{R}$ is the 1-Lipschitz function, from the definition of the observable diameter, there is an interval $A = [s, t] \subseteq [0, +\infty)$ such that $\text{diam } A \leq \text{ObsDiam}_T(X; -\kappa)$ and $(g \circ f)_*(\mu_X)(A) \geq m - \kappa$. Observe that the set $g^{-1}(A)$ is the annulus $\{z \in T \mid s \leq d_T(z, z_0) \leq t\}$. We denote by \mathcal{C} the set of all connected components of the set $g^{-1}(A) \setminus \{z_0\}$.

Claim 5.2. *Assume that $s > 0$. Then, for any $T' \in \mathcal{C}$, we have $\text{diam } T' \leq 2 \text{diam } A$.*

Proof. Given any $z_1, z_2 \in T'$, we shall show that $\phi_{z_0, z_1}(s) = \phi_{z_0, z_2}(s)$. Suppose that $\phi_{z_0, z_1}(s) \neq \phi_{z_0, z_2}(s)$. Then, putting $s_0 := \sup\{t \in [0, +\infty) \mid \phi_{z_0, z_1}(t) = \phi_{z_0, z_2}(t)\}$, we have $s_0 < s$. From the definition of s_0 and the property (2) of \mathbb{R} -trees, we have $(\phi_{z_0, z_1}(s_0), z_1]_T \cap (\phi_{z_0, z_2}(s_0), z_2]_T = \emptyset$. Therefore, from the property (2) of \mathbb{R} -trees, we get

$$[z_1, z_2]_T = [\phi_{z_0, z_1}(s_0), z_1]_T \cup [\phi_{z_0, z_2}(s_0), z_2]_T.$$

Hence, since T' is convex by virtue of Proposition 2.9, the points z_1 and z_2 must be included in different components in $\mathcal{C}_T(\phi_{z_0, z_1}(s_0))$. This is a contradiction, since $T' = \bigcup \{C \cap T' \mid C \in \mathcal{C}_T(\phi_{z_0, z_1}(s_0))\}$ and T' is connected. Thus, we have $\phi_{z_0, z_1}(s) = \phi_{z_0, z_2}(s)$. Consequently, we obtain

$$d_T(z_1, z_2) \leq d_T(z_1, \phi_{z_0, z_1}(s)) + d_T(\phi_{z_0, z_2}(s), z_2) \leq 2(t - s) \leq 2 \text{ObsDiam}_{\mathbb{R}}(X; -\kappa).$$

This completes the proof of the claim. \square

Assume first that $s \leq \text{Sep}(X; \kappa/3, \kappa/3)/2$. Since every path connecting two components in \mathcal{C} must cross the point z_0 , by Claim 5.2, we have

$$\text{diam}(f_*(\mu_X), m - \kappa) \leq \text{diam } g^{-1}(A) \leq \text{Sep} \left(X; \frac{\kappa}{3}, \frac{\kappa}{3} \right) + 4 \text{ObsDiam}_{\mathbb{R}}(X; -\kappa).$$

We consider the other case that $s > \text{Sep}(X; \kappa/3, \kappa/3)/2$. Suppose that $f_*(\mu_X)(T') < \kappa/3$ for any $T' \in \mathcal{C}$. Since $f_*(\mu_X)(g^{-1}(A)) \geq m - \kappa \geq \kappa$, we have $\mathcal{C}' \subseteq \mathcal{C}$ such that

$$\frac{\kappa}{3} \leq f_*(\mu_X) \left(\bigcup \mathcal{C}' \right) < \frac{2\kappa}{3}.$$

Hence, by putting $\mathcal{C}'' := \mathcal{C} \setminus \mathcal{C}'$, we get

$$\text{Sep} \left(X; \frac{\kappa}{3}, \frac{\kappa}{3} \right) < d_T \left(\bigcup \mathcal{C}', \bigcup \mathcal{C}'' \right) \leq \text{Sep} \left(f_*(\mu_X); \frac{\kappa}{3}, \frac{\kappa}{3} \right) \leq \text{Sep} \left(X; \frac{\kappa}{3}, \frac{\kappa}{3} \right),$$

which is a contradiction. Thereby, there exists $T' \in \mathcal{C}$ such that $f_*(\mu_X)(T') \geq \kappa/3$. For a subset $A \subseteq T$ and $r > 0$, we put $A_r := \{z \in T \mid d_T(z, A) \leq r\}$.

Claim 5.3. $f_*(\mu_X)((T')_{\text{Sep}(X;\kappa/3,\kappa/3)}) \geq m - 2\kappa/3$.

Proof. Suppose that $f_*(\mu_X)((T')_{\text{Sep}(X;\kappa/3,\kappa/3)}) < m - 2\kappa/3$. Then, we have a contradiction since

$$\text{Sep}\left(X; \frac{\kappa}{3}, \frac{\kappa}{3}\right) < d_T(T', T \setminus (T')_{\text{Sep}(X;\kappa/3,\kappa/3)+\varepsilon}) \leq \text{Sep}\left(f_*(\mu_X); \frac{\kappa}{3}, \frac{\kappa}{3}\right) \leq \text{Sep}\left(X; \frac{\kappa}{3}, \frac{\kappa}{3}\right)$$

for any sufficiently small $\varepsilon > 0$. \square

Combining Claims 5.2 with 5.3, we obtain

$$\text{diam}(f_*(\mu_X), m - \kappa) \leq \text{diam}((T')_{\text{Sep}(X;\kappa/3,\kappa/3)}) \leq 2 \text{Sep}\left(X; \frac{\kappa}{3}, \frac{\kappa}{3}\right) + 2 \text{ObsDiam}_{\mathbb{R}}(X; -\kappa)$$

and so (5.3). This completes the proof of the theorem. \square

Note that the inequality (5.3) yields slightly worse estimate for the observable diameter $\text{ObsDiam}_T(X; -\kappa)$ than (5.2).

Let T be an \mathbb{R} -tree and $\nu \in \mathcal{B}^1(T)$ with $m := \nu(X)$. Taking a median $m_\nu \in T$ of the measure ν , we let T_ν an element in $\mathcal{C}'_T(c(\nu))$ with $m_\nu \in T_\nu$. We then define the function $\varphi_\nu : T \rightarrow \mathbb{R}$ by $\varphi_\nu(w) := d_T(z, w)$ if $w \in T_\nu$ and $\varphi_\nu(w) := -d_T(z, w)$ otherwise. The function φ_ν is clearly the 1-Lipschitz function.

Lemma 5.4. *Let T be an \mathbb{R} -tree and $\nu \in \mathcal{B}^1(T)$. Then, the function $\varphi_\nu : T \rightarrow \mathbb{R}$ satisfies that $c((\varphi_\nu)_*(\nu)) \leq 0$,*

$$(5.4) \quad |c((\varphi_\nu)_*(\nu))| \leq \text{CRad}((\varphi_\nu)_*(\nu), m - \kappa) + \text{Sep}\left((\varphi_\nu)_*(\nu); \frac{m}{3}, \frac{\kappa}{2}\right) + \text{Sep}((\varphi_\nu)_*(\nu); m - \kappa, m - \kappa),$$

and

$$(5.5) \quad \text{CRad}(\nu, m - \kappa) \leq \text{CRad}((\varphi_\nu)_*(\nu), m - \kappa) + \text{Sep}\left(\nu; \frac{m}{3}, \frac{\kappa}{2}\right) + \text{Sep}\left((\varphi_\nu)_*(\nu); \frac{m}{3}, \frac{\kappa}{2}\right) + \text{Sep}((\varphi_\nu)_*(\nu); m - \kappa, m - \kappa)$$

for any $\kappa > 0$.

Proof. Combining Lemma 2.11 with Proposition 4.1, we have

$$\begin{aligned} \nu(T)c((\varphi_\nu)_*(\nu)) &= \int_T \varphi_\nu(z) d\nu(z) = \int_{T_\nu} d_T(c(\nu), z) d\nu(z) - \int_{T \setminus T_\nu} d_T(c(\nu), z) d\nu(z) \\ &= c_{T_\nu, c(\nu)}(\nu) \leq 0. \end{aligned}$$

Put $r_1 := \text{CRad}((\varphi_\nu)_*(\nu), m - \kappa)$ and $r_2 := \text{Sep}((\varphi_\nu)_*(\nu); m/3, \kappa/2)$. From (5.1), we observe that $(\varphi_\nu)_*(\nu)(B_{\mathbb{R}}(\varphi_\nu(m_\nu), r_2)) \geq \nu(B_T(m_\nu, r_2)) \geq m - \kappa$. Thus, we get

$$(5.6) \quad d_{\mathbb{R}}(B_{\mathbb{R}}(c((\varphi_\nu)_*(\nu)), r_1), B_{\mathbb{R}}(\varphi_\nu(m_\nu), r_2)) \leq \text{Sep}((\varphi_\nu)_*(\nu); m - \kappa, m - \kappa)$$

and so (5.4). The above inequality (5.6) together with $c((\varphi_\nu)_*(\nu)) \leq 0$ yields that

$$\begin{aligned} d_T(c(\nu), m_\nu) &= \varphi_\nu(m_\nu) \leq |c((\varphi_\nu)_*(\nu)) - \varphi_\nu(m_\nu)| \\ &\leq r_1 + r_2 + \text{Sep}((\varphi_\nu)_*(\nu); m - \kappa, m - \kappa) =: r_3. \end{aligned}$$

Therefore, putting $r_4 := \text{Sep}(\nu; m/3, \kappa/2)$, we obtain

$$\nu(B_T(c(\nu), r_3 + r_4)) \geq \nu(B_T(m_\nu, r_4)) \geq m - \kappa$$

and so (5.5). This completes the proof. \square

Proposition 5.5. *Let T be an \mathbb{R} -tree and X an mm-space with $\mu_X \in \mathcal{B}^1(X)$. Then, for any $\kappa > 0$ we have*

$$\text{ObsCRad}_T(X; -\kappa) \leq \text{ObsCRad}_{\mathbb{R}}(X; -\kappa) + 2 \text{Sep}\left(X; \frac{m}{3}, \frac{\kappa}{2}\right) + \text{Sep}(X; m - \kappa, m - \kappa).$$

Proof. This follows from Lemma 2.2 and Lemma 5.4. \square

Proof of Theorem 1.2. Proposition 5.5 together with Corollary 2.6 and Lemma 2.16 directly implies the proof of the theorem. \square

Lemma 5.6. *Let T be an \mathbb{R} -tree and $\nu \in \mathcal{B}^1(T)$. Then, for any $p \geq 1$ and $\kappa > 0$, we have*

$$(5.7) \quad V_p(\nu) \leq 2m^{2/p} \left\{ \text{CRad}((\varphi_\nu)_*(\nu), m - \kappa) + \text{Sep}\left((\varphi_\nu)_*(\nu); \frac{m}{3}, \frac{\kappa}{2}\right) + \text{Sep}((\varphi_\nu)_*(\nu); m - \kappa, m - \kappa) \right\} + 2V_p(\varphi_\nu).$$

In the case of $p = 2$, we also have the better estimate

$$(5.8) \quad V_2(\nu)^2 \leq 4m^2 \left\{ \text{CRad}((\varphi_\nu)_*(\nu), m - \kappa) + \text{Sep}\left((\varphi_\nu)_*(\nu); \frac{m}{3}, \frac{\kappa}{2}\right) + \text{Sep}((\varphi_\nu)_*(\nu); m - \kappa, m - \kappa) \right\}^2 + 2V_2(\varphi_\nu)^2.$$

Proof. From the triangle inequality, we have

$$(5.9) \quad V_p(\nu) \leq 2 \left(\int \int_{T \times T} d_T(c(\nu), z)^p d\nu(z) d\nu(w) \right)^{1/p} = 2 \left(m \int_T d_T(c(\nu), z)^p d\nu(z) \right)^{1/p}.$$

Putting $c_\nu := c((\varphi_\nu)_*(\nu))$, we also get

$$(5.10) \quad \begin{aligned} \left(\int_T d_T(c(\nu), z)^p d\nu(z) \right)^{1/p} &= \left(\int_T |\varphi_\nu(z)|^p d\nu(z) \right)^{1/p} \\ &\leq m^{1/p} |c_\nu| + \left(\int_{\mathbb{R}} |c_\nu - r|^p d(\varphi_\nu)_*(\nu)(r) \right)^{1/p} \\ &\leq m^{1/p} |c_\nu| + \frac{V_p(\varphi_\nu)}{m^{1/p}}, \end{aligned}$$

where in the last inequality we used Lemma 2.11. Combining (5.9) with (5.10), we obtain (5.7).

In the case of $p = 2$, we have

$$\begin{aligned}
 (5.11) \quad \int_T d_T(c(\nu), z)^2 d\nu(z) &= \int_{\mathbb{R}} |r|^2 d(\varphi_\nu)_*(\nu)(r) \\
 &= m|c_\nu|^2 + \int_{\mathbb{R}} |r - c_\nu|^2 d(\varphi_\nu)_*(\nu)(r) \\
 &= m|c_\nu|^2 + \frac{V_2(\varphi_\nu)^2}{2m},
 \end{aligned}$$

where in the second and the last equalities we used Lemma 2.11. Substituting (5.11) to (5.9), we obtain (5.8). This completes the proof. \square

Proposition 5.7. *Let T be an \mathbb{R} -tree and X an mm-space. Then, for any $p \geq 1$, we have*

$$(5.12) \quad \text{Obs}L^p\text{-Var}_T(X) \leq 2\{2^{1/p}(1 + 2 \cdot 2^{1/p}) + 1\} \text{Obs}L^p\text{-Var}_{\mathbb{R}}(X).$$

In the case of $p = 2$, we also have the better estimate

$$(5.13) \quad \text{Obs}L^2\text{-Var}_T(X)^2 \leq (38 + 16\sqrt{2}) \text{Obs}L^2\text{-Var}_{\mathbb{R}}(X)^2.$$

Proof. Assume first that $f_*(\mu_X) \in \mathcal{B}^1(T)$ for any 1-Lipschitz map $f : X \rightarrow T$. Then, Lemma 2.2 together with Lemma 2.3 and (5.7) implies that

$$\begin{aligned}
 \text{Obs}L^p\text{-Var}_T(X) &\leq 2m^{2/p} \left\{ \text{ObsCRad}_{\mathbb{R}}(X; -\kappa) + \text{Sep} \left(X; \frac{m}{3}, \frac{\kappa}{2} \right) \right\} + 2 \text{Obs}L^p\text{-Var}_{\mathbb{R}}(X) \\
 &\leq 2m^{2/p} \left\{ \text{ObsCRad}_{\mathbb{R}}(X; -\kappa) + \text{Sep} \left(X; \frac{\kappa}{2}, \frac{\kappa}{2} \right) \right\} + 2 \text{Obs}L^p\text{-Var}_{\mathbb{R}}(X)
 \end{aligned}$$

for any $0 < \kappa < m/2$. Hence, applying the inequalities (2.3) and (2.5) to this inequality, we get

$$\text{Obs}L^p\text{-Var}_T(X) \leq 2\{m^{1/p}\kappa^{-1/p}(1 + 2 \cdot 2^{1/p}) + 1\} \text{Obs}L^p\text{-Var}_{\mathbb{R}}(X)$$

for any $0 < \kappa < m/2$. Letting $\kappa \rightarrow m/2$, we get (5.12). In the case of $p = 2$, from (5.8), we have

$$\text{Obs}L^2\text{-Var}_T(X)^2 \leq 4m^2 \left\{ \text{ObsCRad}_{\mathbb{R}}(X; -\kappa) + \text{Sep} \left(X; \frac{\kappa}{2}, \frac{\kappa}{2} \right) \right\}^2 + 2 \text{Obs}L^2\text{-Var}_{\mathbb{R}}(X)^2$$

for any $0 < \kappa < m/2$. Therefore, substituting the inequalities (2.4) and (2.6) to this inequality, we get

$$\text{Obs}L^2\text{-Var}_T(X)^2 \leq 2\{m\kappa^{-1}(2\sqrt{2} + 1)^2 + 1\} \text{Obs}L^2\text{-Var}_{\mathbb{R}}(X)^2$$

for any $0 < \kappa < m/2$. Letting $\kappa \rightarrow m/2$, we obtain (5.13).

We consider the other case that there exists a 1-Lipschitz map $f : X \rightarrow T$ with $f_*(\mu_X) \notin \mathcal{B}^1(T)$. By using Hölder's inequality and Fubini's theorem, we have $V_p(f) = +\infty$. Taking $x_0 \in X$, we put $f_n := f|_{B_X(x_0, n)}$ for each $n \in \mathbb{N}$. From Lemma 2.7 and the above proof, we have

$$\begin{aligned}
 V_p(f_n) &\leq \text{Obs}L^p\text{-Var}_T(B_X(x_0, n)) \leq 2\{2^{1/p}(1 + 2 \cdot 2^{1/p}) + 1\} \text{Obs}L^p\text{-Var}_{\mathbb{R}}(B_X(x_0, n)) \\
 &\leq 2\{2^{1/p}(1 + 2 \cdot 2^{1/p}) + 1\} \text{Obs}L^p\text{-Var}_{\mathbb{R}}(X).
 \end{aligned}$$

Since $V_2(f_n) \rightarrow V_2(f) = +\infty$ as $n \rightarrow \infty$, this implies $\text{Obs}L^p\text{-Var}_{\mathbb{R}}(X) = +\infty$. This completes the proof. \square

Proof of Theorem 1.3. Proposition 5.7 directly implies the proof of the theorem.. \square

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